

C_0 -discrete subgroups of $\text{Diff}(I)$ and Weak Margulis Lemma

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ABSTRACT: We prove a weaker version of Margulis Lemma for subgroups of $\text{Diff}(I)$. We also show that a group with commutator subgroup containing a free subsemigroup does not admit a C_0 -discrete faithful representation in $\text{Diff}(I)$.

In this paper, we continue our study of discrete subgroups of $\text{Diff}_+(I)$ - the group of orientation preserving diffeomorphisms of the closed interval $I = [0, 1]$. Following recent trends, we try to view the group $\text{Diff}_+(I)$ as an analog of a Lie group, and we study still basic questions about discrete subgroups of it. This paper can be viewed as a continuation of [A] although the proofs of the results of this paper are independent of [A].

Throughout the paper, the letter G will denote the group $\text{Diff}_+(I)$. On G , we assume the metric induced by the standard norm of the Banach space $C^1[0, 1]$. We will denote this metric by d_1 . Sometimes, we also will consider the metric in G which comes from the standard sup norm $\|f\|_0 = \sup_{x \in [0, 1]} |f(x)|$ of $C[0, 1]$ which we will denote by d_0 . However, unless specified, the metric in all the groups $\text{Diff}_+^r(I)$, $r \in \mathbb{R}$, $r \geq 1$ will be assumed to be d_1 .

The central theme of the paper is the Margulis Lemma. This lemma states that in a connected Lie group H there exists an open non-empty neighborhood U of identity such that any discrete subgroup generated by elements from U is nilpotent. For example, if H is a simple Lie group (such as $SL_2(\mathbb{R})$), and $\Gamma \leq H$ is a lattice, then Γ cannot be generated by elements too close to identity.

In this paper we prove weak versions of Margulis Lemma for the group $G = \text{Diff}_+(I)$. Our study leads us to showing that finitely generated groups with exponential growth which satisfy a very mild condition do not admit faithful C_0 -discrete representation in G :

Theorem A. Let Γ be a subgroup of G , and $f, g \in [\Gamma, \Gamma]$ such that f and g generate a free subsemigroup. Then Γ is not C_0 -discrete.

We also study the Margulis Lemma for the relatives of G such as $Diff_+^{1+c}(I)$, $c \in \mathbb{R}$, $c > 0$ - the group of orientation preserving diffeomorphisms of regularity $1 + c$. In the case of $Diff_+^{1+c}[0, 1]$, we show that there is a more severe rarity of C_0 -discrete subgroups.

Theorem B. Let Γ be a C_0 -discrete subgroup of $Diff_+^{1+c}[0, 1]$. Then Γ is solvable with solvability degree at most $k(c)$.

Theorem B can be strengthened if the regularity is increased further

Theorem C. If Γ is C_0 -discrete subgroup of $Diff_+^2[0, 1]$ then Γ is metaabelian.

It follows from the results of [A], as remarked there, that Margulis Lemma does not hold neither for $Diff_+(I)$ nor for $Homeo_+(I)$, in metrics d_1 and d_0 respectively.

In the increased regularity the lemma still fails: given an arbitrary open neighborhood U of identity diffeomorphism in G , it is easy to find two C^∞ "bump functions" in U which generate a discrete group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$; thus the lemma fails for $Diff_+^\infty(I)$.

Because of the failure of the lemma, it is natural to consider strongly discrete subgroups which we have defined in [A]. Indeed, for strongly discrete subgroups, we are able to obtain positive results which are natural substitutes for Margulis Lemma.

Let us recall the definition of strongly discrete representations from [A]: let Γ be a group, and $\pi : \Gamma \rightarrow Diff_+(I)$ be a faithful discrete representation of it with respect to d_1 metric.

Definition 1. The representation π is called *strongly discrete* if there exists $C > 0$ and $x_0 \in (0, 1)$ such that $|\pi(g)'(x_0) - 1| > C$ for all $g \in \Gamma \setminus \{1\}$. Similarly, we say π is *C_0 -strongly discrete* if $|\pi(g)(x_0) - x_0| > C$ for all $g \in \Gamma \setminus \{1\}$.

Let us note that a strongly discrete subgroup of G is discrete; and C_0 -strongly discrete subgroup of G is C_0 -discrete.

For convenience of the reader, let us recall several basic notions and notations about the growth of groups: if Γ is a finitely generated group, and S a finite generating set, we will write $\omega(\Gamma, S) = \lim_{n \rightarrow \infty} \sqrt[n]{|B_n(1; S, \Gamma)|}$ where $B_n(1; S, \Gamma)$ denotes the ball of radius n around the identity element. (Often we will denote this ball simply by $B_n(1)$). We will also write $\omega(\Gamma) = \inf_{|S| < \infty, \langle S \rangle = \Gamma} \omega(\Gamma, S)$ where the infimum is taken over all finite generating sets S of Γ . If $\omega(\Gamma) > 1$ then one says that Γ has a uniform exponential growth.

Now we are ready to state weak versions of Margulis Lemma for the group G . First, we state a theorem about C_0 -strongly discrete subgroups.

Theorem 1. Let $\omega > 1$. Then there exists an open non-empty neighborhood U of identity $1 \in \text{Diff}_+^1[0, 1]$ such that if Γ is a finitely generated C_0 -strongly discrete subgroup of $\text{Diff}_+^1[0, 1]$ with $\omega(\Gamma) \geq \omega$, then Γ cannot be generated by elements from U .

By increasing the regularity, we can prove a similar version for strongly discrete subgroups

Theorem 2. Let $\omega > 1$. Then there exists an open non-empty neighborhood U of identity $1 \in \text{Diff}_+^1[0, 1]$ such that if Γ is a finitely generated strongly discrete subgroup of $\text{Diff}_+^2[0, 1]$ with $\omega(\Gamma) \geq \omega$, then Γ cannot be generated by elements from U .

Remark 1. In regard to Margulis Lemma, it is interesting to ask a reverse question, i.e. given an arbitrary open neighborhood U of identity in G , is it true that any finitely generated torsion free nilpotent group Γ admits a faithful discrete representation in G generated by elements from U ? In [FF], it is proved that any such Γ does admit a faithful representation into G generated by diffeomorphisms from U . Also, it is proved in [N1] that any finitely generated nilpotent subgroup of G indeed can be conjugated to a subgroup generated by elements from U .

Remark 2. Because of the assumptions about uniform exponential growth in Theorem 1 and Theorem 2, it is natural to ask whether or not every finitely generated subgroup of G of exponential growth has a uniformly exponential growth. This question has already been raised in [N2].

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Proofs of Theorem 1 and Theorem 2

In this section we prove Theorems 1 and 2.

Proof of Theorem 1. We can choose $\lambda > 1$ such that $\lambda < \omega(\Gamma)$. Then the cardinality of the sphere of radius n of Γ with respect to any fixed finite generating set is bigger than the exponential function λ^n , for infinitely many n .

Then let $\epsilon > 0$ such that $(1 - 10\epsilon)\lambda > 1$. We let U be the ϵ -neighborhood of identity in G . (with respect to d_1 metric, we always assume d_1 metric in G unless otherwise stated).

Let Γ be generated by finitely many non-trivial diffeomorphisms $f_1, f_2, \dots, f_s \in U$. We fix this generating set and denote it by S , i.e. $S = \{f_1, f_1^{-1}, \dots, f_s, f_s^{-1}\}$.

We want to prove that Γ is not C_0 -strongly discrete. Assuming the opposite, let $x_0 \in (0, 1)$ such that for some $C > 0$, $|\pi(g)(x_0) - x_0| > C$ for all $g \in \Gamma \setminus \{1\}$.

Let $B_n(1)$ be the ball of radius n around identity in the Cayley graph of Γ with respect to S . Then $\text{Card}(B_n(1) \setminus B_{n-1}(1)) > \lambda^n$ for infinitely many $n \in \mathbb{N}$. Let A denotes the set of all such n .

Let Δ be a closed subinterval of $(0, 1)$ of length less than C such that x_0 is the left end of Δ .

We denote the right-invariant Cayley metric of Γ with respect to S by $|\cdot|$. For all $g \in \Gamma$, let $\Delta_g = g(\Delta)$. Thus we have a collection $\{\Delta_g\}_{g \in G}$ of closed subintervals of $(0, 1)$.

Notice that if $g = sw$, $s \in S$ then by Mean Value Theorem, $|\Delta_{sw}| > (1 - 10\epsilon)|s(\Delta_w)|$. Then, necessarily, for all $n \in A$, we have $\sum_{|g|=n} |\Delta_g| > (1 - 10\epsilon)^n \lambda^n |\Delta| \rightarrow \infty$ as $n \rightarrow \infty$.

Then $\exists g_1, g_2 \in \Gamma$, $g_1 \neq g_2$ such that $g_2(x_0) \in \Delta_{g_1}$. Then $g_1^{-1}g_2(x_0) \in \Delta$. Since $|\Delta| < C$ we obtain a contradiction. \square

Now we prove a better result by assuming higher regularity for the representation

Proof of Theorem 2. Let $\lambda > 1$ be a constant such that $\lambda < \omega(\Gamma)$. Then the cardinality of the sphere of radius n of Γ with respect to any fixed finite generating set is bigger than the exponential function λ^n , for infinitely many n .

We choose $\epsilon > 0, \eta > 0$ to be such that $1 < \eta < \frac{\lambda}{1+\epsilon}$ and $(1 - 10\epsilon)\eta > 1$. Let U be the ball of radius ϵ around the identity diffeomorphism.

We again assume Γ is generated by finitely many non-trivial diffeomorphisms $f_1, f_2, \dots, f_s \in U$, and we fix the generating set $S = \{f_1, f_1^{-1}, \dots, f_s, f_s^{-1}\}$. Let $B_n(1)$ be the ball of radius n around identity in the Cayley graph of Γ with respect to S . Then $\text{Card}(B_n(1) \setminus B_{n-1}(1)) > \lambda^n$ for infinitely many $n \in \mathbb{N}$. Let A denotes the set of all such n .

We need to show that Γ is not strongly discrete. Assuming the opposite, let $x_0 \in (0, 1)$ such that for some $C > 0$, $|\pi(g)'(x_0) - 1| > C$ for all $g \in \Gamma \setminus \{1\}$.

Let C_1 be a positive number such that $1 - C < (1 - C_1)^2$, $1 + C > (1 + C_1)^2$. Let also $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$(1 + \frac{1}{\eta^n})^n < 1 + C_1$$

Notice that for all $n \in A$, $g \in B_n(1) \setminus B_{n-1}(1)$, and $x \in [0, 1]$, we have $(1 - \epsilon)^n < g'(x) < (1 + \epsilon)^n$. Since $1 + \epsilon < \lambda$ and $\eta < \lambda$, by the argument of the proof of Theorem 1, there exists $n \in A$ and $g_1, g_2 \in \Gamma$ such that

$$n > N_1, \quad g_1 \neq g_2, \quad |g_1| = |g_2| = n$$

but

$$|g_1(x_0) - g_2(x_0)| \leq \frac{1}{\eta^n}, \quad 1 - C_1 < \frac{g_1'(x_0)}{g_2'(x_0)} < 1 + C_1$$

Let $y_0 = g_1(x_0)$, $z_0 = g_2(x_0)$, $W = g_1^{-1}$, $V = g_1^{-1}g_2$, and let $W = h_n h_{n-1} \dots h_1$ where W is a reduced word in the alphabet S of length n and $h_i \in S$, $1 \leq i \leq n$.

Let also W_k be the suffix of W of length k , $y_k = W_k(y_0)$, $z_k = W_k(z_0)$, $1 \leq k \leq n$.

Furthermore, let $\max_{1 \leq i \leq s} \sup_{0 \leq y \neq z \leq 1} \frac{|f_i'(y) - f_i'(z)|}{|y - z|} = M$, and $L = 1 + \epsilon$.

Then we have

$$|y_k - z_k| \leq \frac{L^k}{\lambda^n}, \quad |h'_{k+1}(y_k) - h'_{k+1}(z_k)| \leq \frac{ML^k}{\lambda^n}, \quad 0 \leq k \leq n - 1$$

Then $1 - \frac{ML^{k+1}}{\lambda^n} \leq \frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)} \leq 1 + \frac{ML^{k+1}}{\lambda^n}$ for all $0 \leq k \leq n - 1$. From here we obtain that

$$\prod_{k=0}^{n-1} (1 - \frac{ML^{k+1}}{\lambda^n}) \leq \prod_{k=0}^{n-1} \frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)} \leq \prod_{k=0}^{n-1} (1 + \frac{ML^{k+1}}{\lambda^n})$$

Then, for sufficiently big n in A

$$(1 - \frac{1}{\eta^n})^n = \prod_{k=0}^{n-1} (1 - \frac{1}{\eta^n}) \leq \prod_{k=0}^{n-1} \frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)} \leq \prod_{k=0}^{n-1} (1 + \frac{1}{\eta^n}) = (1 + \frac{1}{\eta^n})^n$$

Since $\prod_{k=0}^{n-1} \frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)} = \frac{(g_1^{-1})'(y_0)}{(g_1^{-1})'(z_0)}$ we obtain that $1 - C_1 < \frac{(g_1^{-1})'(y_0)}{(g_1^{-1})'(z_0)} < 1 + C_1$.

Then

$$V'(x_0) = (g_1^{-1})'(g_2(x_0))g_2'(x_0) = \frac{(g_1^{-1})'(g_2(x_0))g_2'(x_0)}{(g_1^{-1})'(g_1(x_0))g_1'(x_0)}(g_1^{-1})'(g_1(x_0))g_1'(x_0) = \frac{(g_1^{-1})'(g_2(x_0))g_2'(x_0)}{(g_1^{-1})'(g_1(x_0))g_1'(x_0)} \in ((1-C_1)^2, (1+C_1)^2) \subset (1-C, 1+C)$$

Thus we proved that $1 - C < V'(x_0) < 1 + C$ which contradicts our assumption. \square

Remark 3. The same proof, with slight changes, works for representations of C^{1+c} -regularity for any real $c > 0$.

Proofs of Theorems A, B, C

In the proofs of Theorem 1 and of Theorem 2, we consider the orbit of the point x_0 under the action of Γ . By using exponential growth, we find two distinct elements g_1, g_2 such that $g_1(x_0)$ and $g_2(x_0)$ are very close. Then we "pull back" $g_2(x_0)$ by g_1^{-1} , i.e. we consider the point $g_1^{-1}g_2(x_0)$ and show that this point is sufficiently close to x_0 . It is at this stage that we heavily use the condition that Γ is generated by elements from the small neighborhood of $1 \in G$, i.e. derivatives of the generators are uniformly close to 1. However, if Γ is an arbitrary subgroup of the commutator group $[G, G]$, not necessarily close to identity, then for any $x_0 \in (0, 1)$, $f \in \Gamma$ and for any $\epsilon > 0$, there exists $W \in \Gamma$ such that $|f'(W(x_0)) - 1| < \epsilon$; we simply need to find W such that $W(x_0)$ is sufficiently close to 1 (or to 0). This fact provides a new idea of taking x_0 close to 1, then considering the part of the orbit which lies in a small neighborhood of 1, then using exponential growth to find points close to each other in that neighborhood, and then perform the "pull back".

The following proposition is a special case of Theorem A, and answers Question 2 from [A]. For simplicity, we give a separate proof of it.

Proposition 1. \mathbb{F}_2 does not admit a C_0 -discrete representation in G .

Proof. Since the commutator subgroup of \mathbb{F}_2 contains a copy of \mathbb{F}_2 , it is sufficient to prove that \mathbb{F}_2 does not admit a C_0 -discrete representation in $G^{(1)} = [G, G]$.

Let Γ be a subgroup of $G^{(1)}$ isomorphic to \mathbb{F}_2 generated by diffeomorphisms f and g . Without loss of generality we may assume that Γ has no fixed point on $(0, 1)$. Let also $\epsilon > 0$ and $M = \max_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|)$.

We choose $N \in \mathbb{N}, \delta > 0$ such that $1/N < \epsilon$ and for all $x \in [1 - \delta, 1]$, the inequalities $|f'(x) - 1| < 1/10, |g'(x) - 1| < 1/10$ hold.

Let $W = W(f, g)$ be an element of Γ such that $W(1/N) \in [1 - \delta, 1]$, m be the length of the reduced word W . Let also $x_i = i/N, 0 \leq i \leq N$.

For every $n \in \mathbb{N}$, let

$$S_n = \{g \in B_n(1) \mid u(W(x_1)) \geq W(x_1) \text{ for all suffixes } u \text{ of } g\}$$

(Here we view g as a reduced word in the alphabet $\{f, g, f^{-1}, g^{-1}\}$). Then $|S_n| \geq 2^n$.

Then we can choose and fix a sufficiently big n such that the following two conditions are satisfied:

- (i) there exists $g_1, g_2 \in S_n$ such that $g_1 \neq g_2$, and $|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{(1.9)^n}, 1 \leq i \leq N - 1$.
- (ii) $M^m(1.1)^n \frac{1}{(1.9)^n} < \epsilon$.

Let $h_1 = g_1 W, h_2 = g_2 W, y_i = W(x_i), z'_i = g_1(y_i), z''_i = g_2(y_i), 1 \leq i \leq N$.

Then for all $i \in \{1, \dots, N - 1\}$, we have

$$\begin{aligned} |h_1^{-1} h_2(x_i) - x_i| &= |(g_1 W)^{-1}(g_2 W)(x_i) - x_i| = \\ |(g_1 W)^{-1}(g_2 W)(x_i) - (g_1 W)^{-1}(g_1 W)(x_i)| &= |W^{-1} g_1^{-1} g_2(y_i) - W^{-1} g_1^{-1} g_1(y_i)| \\ &= |W^{-1} g_1^{-1}(z''_i) - W^{-1} g_1^{-1}(z'_i)| \end{aligned}$$

Then, since $g_1, g_2 \in S_n$, by Mean Value Theorem, we have $|h_1^{-1} h_2(x_i) - x_i| \leq M^m(1.1)^n |z'_1 - z''_1| < M^m(1.1)^n \frac{1}{(1.9)^n}$. Then, by condition (ii), we obtain $|h_1^{-1} h_2(x_i) - x_i| < \epsilon$. Then $|h_1^{-1} h_2(x) - x| < 2\epsilon$ for all $x \in [0, 1]$. Indeed, let $x \in [x_i, x_{i+1}]$. Then

$$|h_1^{-1} h_2(x) - x| \leq \max\{|h_1^{-1} h_2(x_i) - x|, |h_1^{-1} h_2(x_{i+1}) - x|\}$$

But $|h_1^{-1} h_2(x_i) - x| \leq |h_1^{-1} h_2(x_i) - x_i| + |x_i - x| < 2\epsilon$, and similarly, $|h_1^{-1} h_2(x_{i+1}) - x| \leq |h_1^{-1} h_2(x_{i+1}) - x_{i+1}| + |x_{i+1} - x| < 2\epsilon$.

Since ϵ is arbitrary, we obtain that Γ is not C_0 -discrete. \square

By examining the proof of Proposition 1, we will now prove Theorem A thus obtaining a much stronger result. The inequality $|S_n| > 2^n$ is a crucial fact in the proof of Proposition 1; we need the cardinality of S_n

grow exponentially. If Γ is an arbitrary finitely generated group with an exponential growth, this exponential growth of S_n is not automatically guaranteed. But we can replace S_n by another subset \mathbb{S}_n which still does the job of S_n and which grows exponentially, if we assume a mild condition on Γ .

First we need the following easy

Lemma 1. Let $\alpha, \beta \in G, z_0 \in (0, 1)$ such that $z_0 \leq \alpha(z_0) \leq \beta\alpha(z_0)$. Then $U\beta\alpha(z_0) \geq z_0$ where $U = U(\alpha, \beta)$ is any positive word in letters α, β . \square

Now we are ready to prove Theorem A.

Proof. Without loss of generality, we may assume that Γ has no fixed point on $(0, 1)$. Let again $\epsilon > 0, M = \sup_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|), N \in \mathbb{N}, \delta > 0$ such that $1/N < \epsilon$ and for all $x \in [1 - \delta, 1]$, the inequalities $|f'(x) - 1| < 1/100, |g'(x) - 1| < 1/100$ hold.

Let $W = W(f, g)$ be an element of Γ such that

$$(\{f^i W(1/N) \mid -2 \leq i \leq 2\} \cup \{g^i W(1/N) \mid -2 \leq i \leq 2\}) \subset [1 - \delta, 1]$$

and let m be the length of the reduced word W . Let also $x_i = i/N, 0 \leq i \leq N$ and $z = W(1/N)$.

By replacing the pair (f, g) with (f^{-1}, g^{-1}) if necessary, we may assume that $f(z) \geq z$. Then at least one of the following cases is valid:

Case 1. $f(z) \leq gf(z)$;

Case 2. $z \leq gf(z)$;

Case 3. $gf(z) \leq z$.

If Case 1 holds then we let $\alpha = f, \beta = g, z_0 = z$. If Case 1 does not hold but Case 2 holds, then we let $\alpha = gf, \beta = f, z_0 = z$. Finally, if Case 1 and Case 2 do not hold but Case 3 holds, then we let $\alpha = f^{-1}g^{-1}, \beta = g^{-1}, z_0 = gf(z)$.

In all the three cases, we will have $z_0 \in [1 - \delta, 1], z_0 \leq z$, and α, β generate a free subsemigroup, and conditions of Lemma 1 are satisfied, i.e. we have $z_0 \leq \alpha(z_0) \leq \beta\alpha(z_0)$. Moreover, we notice that $\sup_{0 \leq x \leq 1} (|\alpha'(x)| + |\beta'(x)|) \leq M^2$.

Now, for every $n \in \mathbb{N}$, let

$$\mathbb{S}_n = \{U(\alpha, \beta)\beta\alpha W \mid U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length at most } n.\}$$

Applying Lemma 1 to the pair $\{\alpha, \beta\}$ we obtain that $VW^{-1}(z_0) \geq z_0$ for all $V \in \mathbb{S}_n$.

Then $|\mathbb{S}_n| \geq 2^n$. After achieving this inequality, we proceed as in the proof of Proposition 1 with just a slight change: there exists a sufficiently big n such that the following two conditions are satisfied:

(i) exists $g_1, g_2 \in \mathbb{S}_n$ such that $g_1 \neq g_2$, and $|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{(1.9)^n}$, $1 \leq i \leq N-1$.

(ii) $M^{2m+4}(1.1)^n \frac{1}{(1.9)^n} < \epsilon$.

Let $h_1 = g_1 W, h_2 = g_2 W, y_i = W(x_i), z'_i = g_1(y_i), z''_i = g_2(y_i), 1 \leq i \leq N$.

Then for all $i \in \{1, \dots, N-1\}$, we have $|h_1^{-1} h_2(x_i) - x_i| = |(g_1 W)^{-1}(g_2 W)(x_i) - x_i| = |(g_1 W)^{-1}(g_2 W)(x_i) - (g_1 W)^{-1}(g_1 W)(x_i)| = |W^{-1} g_1^{-1} g_2(y_i) - W^{-1} g_1^{-1} g_1(y_i)| = |W^{-1} g_1^{-1}(z''_i) - W^{-1} g_1^{-1}(z'_i)|$.

Since $g_1, g_2 \in \mathbb{S}_n$, by Mean Value Theorem, we have $|h_1^{-1} h_2(x_i) - x_i| \leq M^{m+2}(1.1)^n |z'_1 - z''_1| < M^{m+2}(1.1)^n \frac{1}{(1.9)^n}$. Since m is fixed, for sufficiently big n we obtain that $|h_1^{-1} h_2(x_i) - x_i| < \epsilon$. Then $|h_1^{-1} h_2(x) - x| < 2\epsilon$ for all $x \in [0, 1]$. Since ϵ is arbitrary, we obtain that Γ is not C_0 -discrete. \square

Proof of Theorem B. Let H be an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$. If H contains a free subsemigroup then we are done by Theorem A. If H does not contain a free subsemigroup then by the result from [N2] H is virtually nilpotent. Then again by the result of [N2], H is solvable of solvability degree at most $l(c)$. Since the natural number $l(c)$ depends only on c , and not on H , and since H is an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$ we obtain that $[\Gamma, \Gamma]$ is solvable of solvability degree at most $l(c)$. Hence Γ is solvable with a solvability degree at most $l(c) + 1$. \square

Proof of Theorem C. Let again H be an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$. Again, if H contains a free subsemigroup then we are done by Theorem A. If H does not contain a free subsemigroup then by the result from [N2] H is virtually nilpotent. Then, by the result of Plante-Thurston ([PT]), H is virtually Abelian. Then, by the result of Szekeres ([S]), H is Abelian. Since H is an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$, we conclude that $[\Gamma, \Gamma]$ is Abelian, hence Γ is metaabelian. \square

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